Revision 5 (Solutions)

Year 11 Examination

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

Student Number: In figures | | | | | | | | |

In words ______________________________

Teacher name ______________________________

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section Two: Calculator-assumed **65% (69 Marks)**

This section has **nine (9)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 70 minutes.

Question 1 **(6 marks)**

The work done, in joules, by a force $\mathbf{F}$ Newtons in changing the displacement of an object $\mathbf{s}$ metres is given by the scalar product of $\mathbf{F}$ and $\mathbf{s}$.

(a) Determine the work done by a force of 200 N that moves an object 2.7 m, given that the force acts at an angle of 17° to the direction of movement. (1 mark)

$$200 \times 2.7 \times \cos 17° = 516.4 \text{ J}$$

(b) When an object is moved $0.8\mathbf{i} - 0.6\mathbf{j}$ m by a force of 130 N, the work done is 126 J.

(i) Show that one possible force is $120\mathbf{i} - 50\mathbf{j}$ N. (2 marks)

$$\begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} \cdot \begin{bmatrix} 120 \\ -50 \end{bmatrix} = 96 + 30 = 126 \text{ J}$$

and

$$\sqrt{120^2 + (-50)^2} = 130 \text{ N}$$

(ii) Another possible force is $x\mathbf{i} + y\mathbf{j}$ N. Determine the values of x and y. (3 marks)

$$\begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 126 \Rightarrow 0.8x - 0.6y = 126$$

$$x^2 + y^2 = 130^2$$

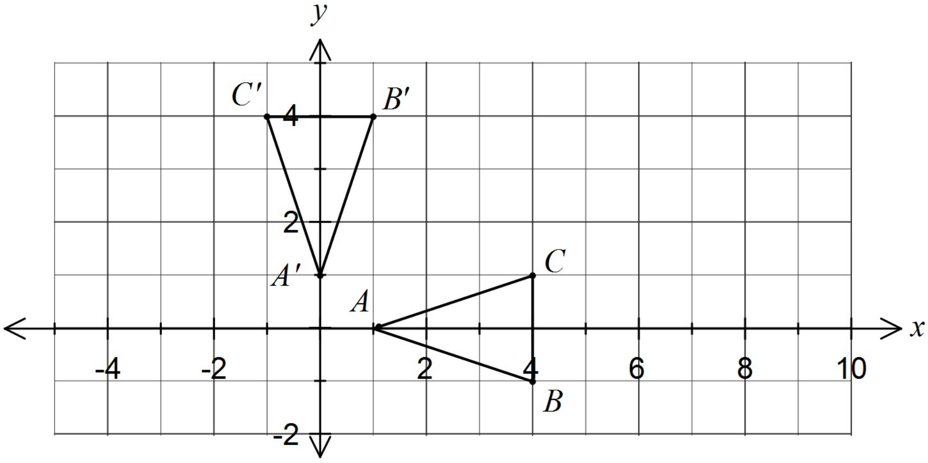
~~$x = 120, y = -50$~~

or

$x = 81.6, y = -101.2$

Question 2 **(7 marks)**

On the axes below, triangle ABC is transformed to $A'B'C'$ by a linear transformation.



(a) State the appropriate transformation matrix. (1 mark)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b) Following a second transformation, $A'(0, 1)$ and $B'(1, 4)$ are transformed to $A''(0, 2)$ and $B''(3, 8)$.

(i) Determine the matrix for this second transformation. (2 marks)

$$T\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 8 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 3 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

(ii) Calculate the area of triangle $A''B''C''$. (2 marks)

$\det(T) = 6$

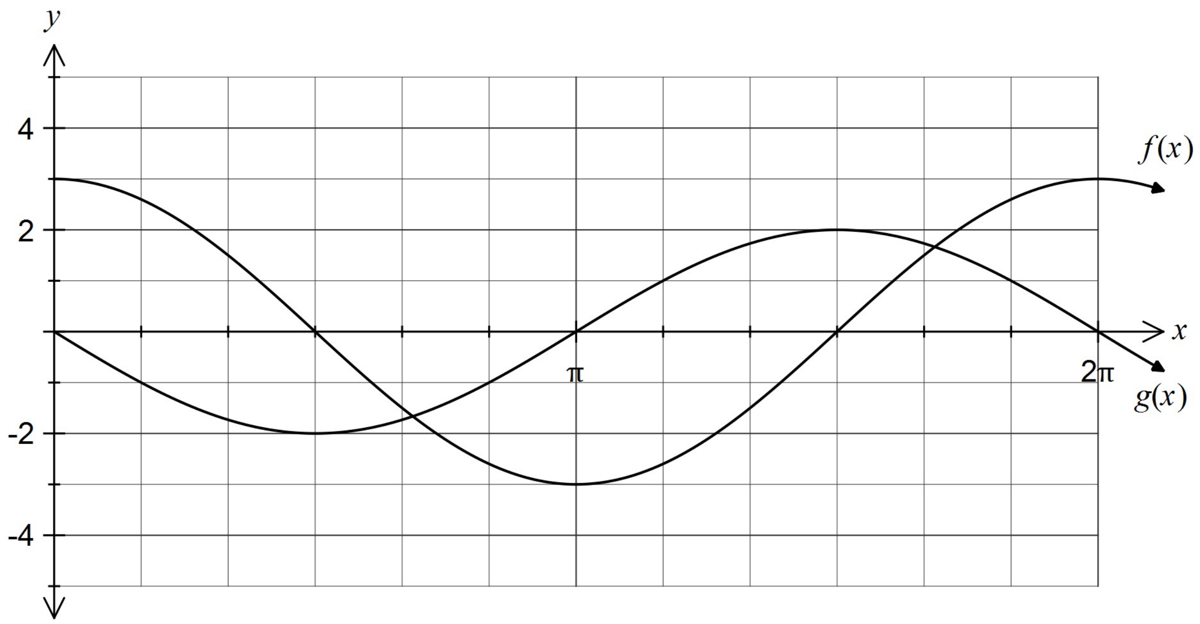
$6 \times 3 = 18$ sq units

(c) Determine the transformation matrix that will transform triangle $A''B''C''$ back to ABC. (2 marks)

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{3} & 0 \end{bmatrix}$$

Question 3 **(9 marks)**

(a) The graphs of $y = f(x)$ and $y = g(x)$ are shown below for $0 \le x \le 2\pi$.



(i) If $f(x) = a\cos x$ and $g(x) = b\sin x$, state the values of a and b. (1 mark)

$a = 3$

$b = -2$

(ii) If $h(x) = f(x) + g(x)$ express $h(x)$ in the form $R\cos(x + \alpha)$. (3 marks)

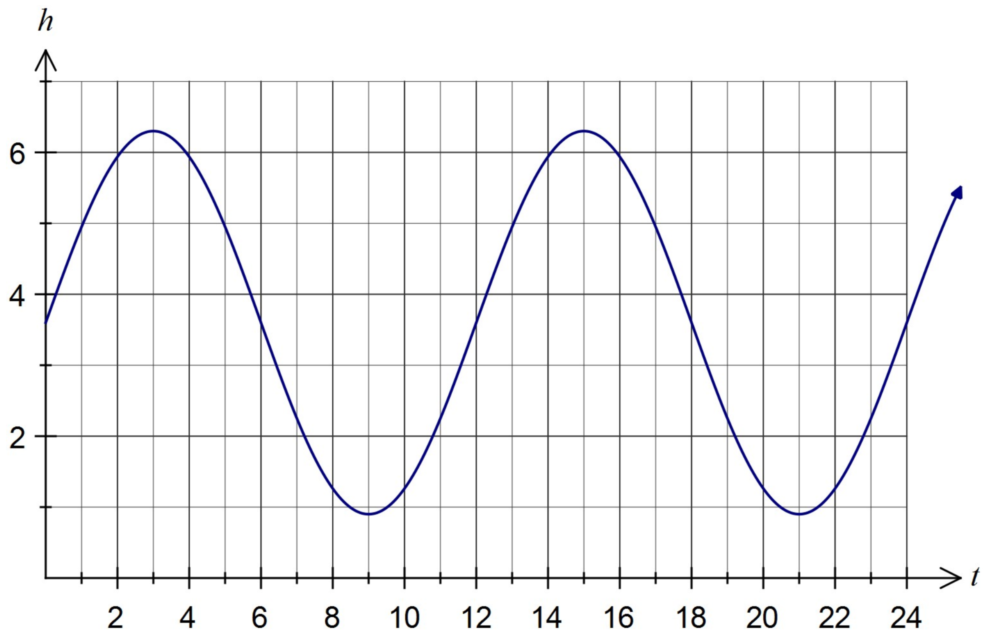
$R = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\alpha = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.588$

$h(x) = \sqrt{13}\cos\left(x + 0.588\right)$

(b) The clearance, h metres, under a bridge spanning a river estuary varies with the time since midnight, t hours, and is given by $h = 3.6 + 2.7\sin\left(\frac{\pi t}{6}\right)$.

(i) Sketch the graph of the clearance against time on the axes below. (3 marks)



(ii) Determine the percentage of any 24-hour period during which the clearance under the bridge is no more than two metres. (2 marks)

$h \le 2 \Rightarrow 7.21 \le t \le 10.79$

$\frac{10.79 - 7.21}{12} \times 100 = 29.8\%$

Question 4 **(8 marks)**

(a) A committee of eight people is to be selected from 10 junior, 14 adult and 11 senior nominations from the members of a club. Determine the number of ways the committee can be selected if

(i) there are no restrictions. (1 mark)

$$^{35}C_8 = 23\ 535\ 820$$

(ii) there must be five adults and more seniors than juniors. (3 marks)

$$^{14}C_5 \times \left(^{11}C_3 \times {}^{10}C_0 + {}^{11}C_2 \times {}^{10}C_1 \right) = 2002 \times (165 \times 1 + 55 \times 10)$$
$$= 1\ 431\ 430$$

(b) Six books are to be selected for promotion in a newsletter from a choice of nine crime, seven fantasy and six romance novels. Determine the number of selections that include three fantasy or three romance novels. (4 marks)

$$n(3F) = {}^{7}C_3 \times {}^{15}C_3 = 15\ 925$$

$$n(3R) = {}^{6}C_3 \times {}^{16}C_3 = 11\ 200$$

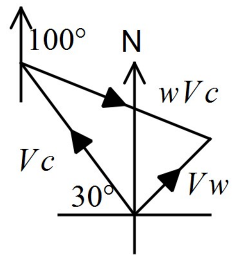
$$n(3F \cap 3R) = {}^{7}C_3 \times {}^{6}C_3 = 700$$

$$n(3F \cup 3R) = 15925 + 11200 - 700 = 26\ 425$$

Question 5 **(7 marks)**

A cyclist pedals at a speed of 25 km/h along a road on a bearing of 300º. Relative to the cyclist, the wind appears to be blowing from 280º with a speed of 30 km/h.

(a) Sketch a labelled diagram to show the relationship between the velocities of the cyclist, the wind and the wind relative to the cyclist. (2 marks)



$$Vw = Vc + wVc$$

(b) Express the velocities of the cyclist and the wind relative to the cyclist in the component form, rounding coefficients to two decimal places. (2 marks)

$$Vc = 25\begin{bmatrix}\cos(150)\\ \sin(150)\end{bmatrix} \approx \begin{bmatrix}-21.65\\ 12.50\end{bmatrix}$$

$$wVc = 30\begin{bmatrix}\cos(-10)\\ \sin(-10)\end{bmatrix} \approx \begin{bmatrix}29.54\\ -5.21\end{bmatrix}$$

(c) Determine the true speed of the wind and the bearing from which it is blowing. (3 marks)

$$Vw = Vc + wVc$$

$$= \begin{bmatrix}-21.65\\ 12.50\end{bmatrix} + \begin{bmatrix}29.54\\ -5.21\end{bmatrix}$$

$$= \begin{bmatrix}7.89\\ 7.29\end{bmatrix}$$

$|Vw| = 10.7$ km/h
at 42.7° from x-axis

Bearing is 047.3°

Or using CAS to solve triangle

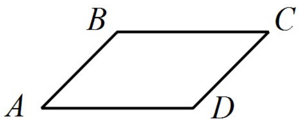
$Vw = 10.7$

at angle of 107.3° to Vc

Bearing is 047.3°

Question 6 **(9 marks)**

(a) Figure $ABCD$ is a parallelogram. Let $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Prove that the diagonals AC and BD are perpendicular only when $|\mathbf{b}| = |\mathbf{d}|$. (4 marks)



$$\mathbf{AC} = \mathbf{b} + \mathbf{d}$$
$$\mathbf{BD} = \mathbf{d} - \mathbf{b}$$

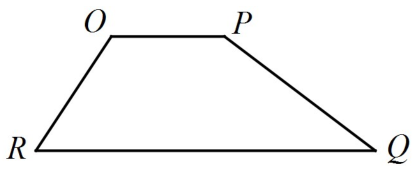
Perpendicular when:

$$\mathbf{AC} \cdot \mathbf{BD} = 0$$
$$(\mathbf{b} + \mathbf{d}) \cdot (\mathbf{d} - \mathbf{b}) = 0$$

$$\mathbf{b} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{b} + \mathbf{d} \cdot \mathbf{d} - \mathbf{d} \cdot \mathbf{b} = 0$$

$$|\mathbf{d}|^2 = |\mathbf{b}|^2 \Rightarrow |\mathbf{d}| = |\mathbf{b}|$$

(b) Figure $OPQR$ is a trapezium, with OP parallel to RQ and $RQ = 3OP$. If M is the point of intersection of OQ and PR, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$, $\overrightarrow{OM} = \lambda\overrightarrow{OQ}$ and $\overrightarrow{RM} = \mu\overrightarrow{RP}$ show that $\overrightarrow{OM} = \frac{1}{4}\mathbf{r} + \frac{3}{4}\mathbf{p}$. (5 marks)



$$\overrightarrow{OM} = \lambda\overrightarrow{OQ} = \overrightarrow{OR} + \mu\overrightarrow{RP}$$

$$\lambda(\mathbf{r} + 3\mathbf{p}) = \mathbf{r} + \mu(\mathbf{p} - \mathbf{r})$$

Equate $\mathbf{r}$ coeffs: $\lambda = 1 - \mu$

Equate $\mathbf{p}$ coeffs: $3\lambda = \mu$

$$\lambda = \tfrac{1}{4}$$

$$\overrightarrow{OM} = \tfrac{1}{4}\overrightarrow{OQ}$$
$$= \tfrac{1}{4}(\mathbf{r} + 3\mathbf{p})$$

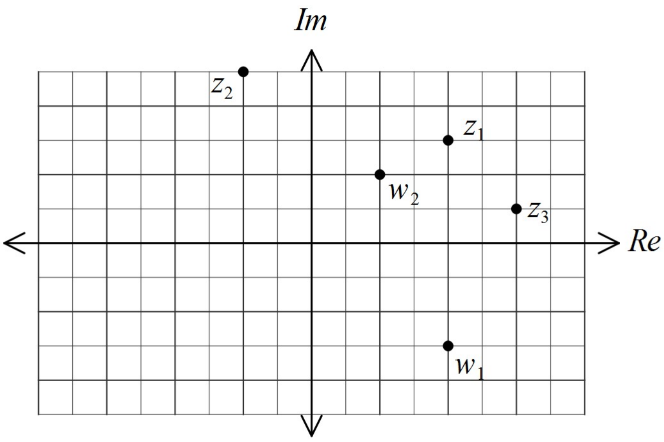
Question 7 **(8 marks)**

(a) Solve $2(z-3)^2+2=0$. (2 marks)

$(z-3)^2=-1=i^2$

$z=3\pm i$

(b) The complex numbers w_1 and w_2 are shown in the Argand plane below.



Plot and label the complex numbers given by

(i) $z_1=\bar{w}_1$. (1 mark)

(ii) $z_2=w_2-w_1$. (1 mark)

(iii) $z_3=\overline{w_1+w_2}$. (1 mark)

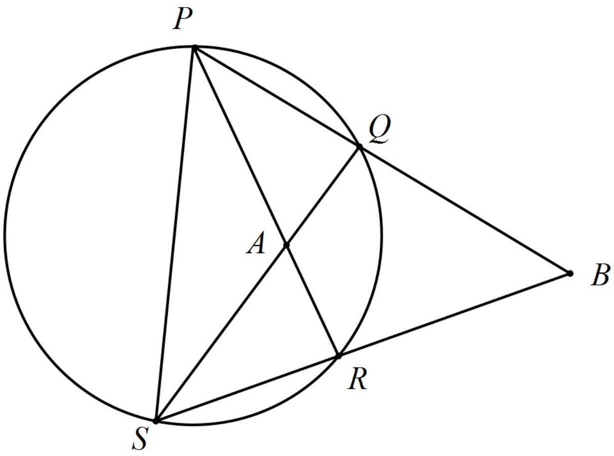
(c) One solution of the quadratic equation $x^2+bx+c=0$ is $x=3-2i$. Determine the values of the real coefficients b and c. (3 marks)

$(x-(3-2i))\times(x-(3+2i))=x^2-6x+13$

$b=-6,\ c=13$

Question 8 **(8 marks)**

The points P, Q, R and S lie on a circle of radius r. PR and QS meet at A. PQ and SR are produced to meet at B, and $AQBR$ is a cyclic quadrilateral.



(a) Prove that BS is perpendicular to PR. (6 marks)

$\angle PQS = 180° - \angle BQS$ (angle on straight line)
$\angle PRS = 180° - \angle PRB$ (angle on straight line)

$\angle PQS = \angle PRS$ (stand on same arc PS)

$180° - \angle BQS = 180° - \angle PRB \Rightarrow \angle BQS = \angle PRB$

$\angle BQS + \angle PRB = 180$ (Opp angles in cyclic quad)

Hence $2\angle PRB = 180 \Rightarrow \angle PRB = 90°$, that is, BS is perpendicular to PR.

(b) Prove that the length of PS is $2r$. (2 marks)

$\angle PRS = \angle PRB = 90°$

Hence PS must be diameter of circle (Angle in semi-circle)

Length of PS is twice radius: $PS = 2r$

Question 9 **(7 marks)**

Let $P(n) = 10^n + 18n - 1$.

(a) If $P(1) = 9a$ and $P(2) = 9b$, evaluate a and b. (2 marks)

$P(1) = 27 = 9 \times 3 \Rightarrow a = 3$

$P(2) = 135 = 9 \times 15 \Rightarrow b = 15$

(b) Prove by induction that $P(n)$ is always a multiple of nine when n is a positive integer. (5 marks)

$P(1) = 9(3)$

When $n = k$

$P(k) = 10^k + 18k - 1 = 9M, \; M \in \mathbb{Z}$

When $n = k + 1$

$$\begin{aligned} P(k+1) &= 10^{k+1} + 18(k+1) - 1 \\ &= 10.10^k + 18k + 18 - 1 \\ &= 10^k + 18k - 1 + 18 + 9.10^k \\ &= 9M + 9(2 + 10^k) \\ &= 9(M + 2 + 10^k) \end{aligned}$$

Thus, as $P(1)$ is true and the truth of $P(k)$ implies the truth of $P(k+1)$, then $P(n)$ is true for all positive integers n.